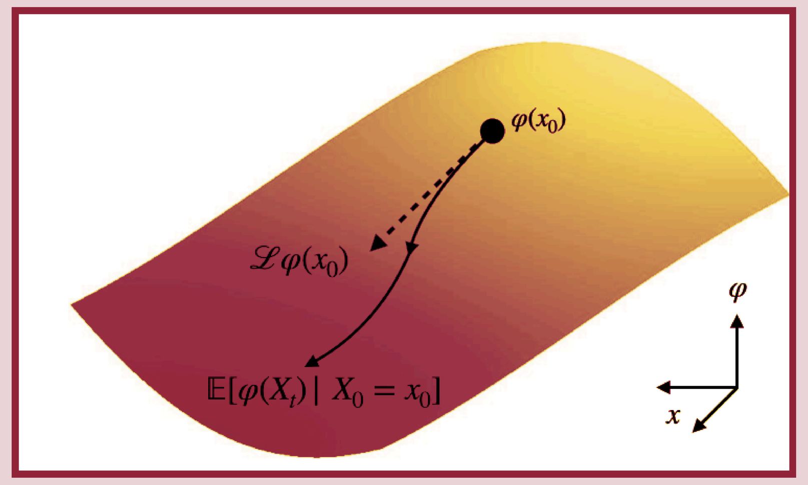
Finite-Data Error Bounds for Approximating the Koopman Operator

Daniel Fassler, Jason Bramburger, Simone Brugiapaglia,
Department of Mathematics and Statistics, Concordia University

Introduction



The Koopman operator is an ubiquitous tool for dynamical system analysis. It permits the analysis of **observable functions** along trajectories. For a Markovian process with timestep τ, it is defined as $\mathcal{K}^\tau\varphi(t,\mathbf{x}) = \mathbb{E}\left[\varphi(t+\tau, \mathbf{x}_{t+\tau}) | \mathbf{x}_\tau = \mathbf{x}\right]$.

The Koopman operator provides an equivalent linear formulation of any dynamical system, through the evolution of **observable functions**.

Extended dynamic mode decomposition

EDMD seeks to approximate the Koopman operator **from data**.

Samples

Data $(\mathbf{x}_i, \mathbf{y}_i)_{i=1}^n$

$\mathbf{y}_i(t) = \mathbf{x}_i(t+\tau)$

Dictionaries

$\boldsymbol{\psi} = (\psi_i)_{i=1}^m$

$\boldsymbol{\phi} = (\phi_i)_{i=1}^l$

EDMD Matrices

$\boldsymbol{\Psi}_n = (\psi_i(\mathbf{x}_j))_{i,j=1}^{m,n}$

$\boldsymbol{\Phi}_n^\tau = (\phi_i(\mathbf{y}_j))_{i,j=1}^{l,n}$

$$\min_{K \in \mathbb{C}^{l \times m}} \|\boldsymbol{\Phi}_n^\tau - K\boldsymbol{\Psi}_n\|_F^2$$

$$K_{mn}^\tau = \boldsymbol{\Phi}_n^\tau(\boldsymbol{\Psi}_n)^\dagger$$

The matrix obtained approximates the Koopman operator on $\mathrm{span}\{\boldsymbol{\phi}\}$ using the basis $\boldsymbol{\psi}$.

Acts by

$$\mathcal{K}_{mn}^\tau\varphi = \mathbf{c} \cdot K_{mn}^\tau \cdot \boldsymbol{\psi}, \quad \varphi = \mathbf{c} \cdot \boldsymbol{\phi}$$

EDMD as a least squares problem

EDMD minimization problem $\longrightarrow$ $\min_{K \in \mathbb{C}^{l \times m}} \|\boldsymbol{\Phi}_n^\tau - K\boldsymbol{\Psi}_n\|_F^2.$

Equivalent to

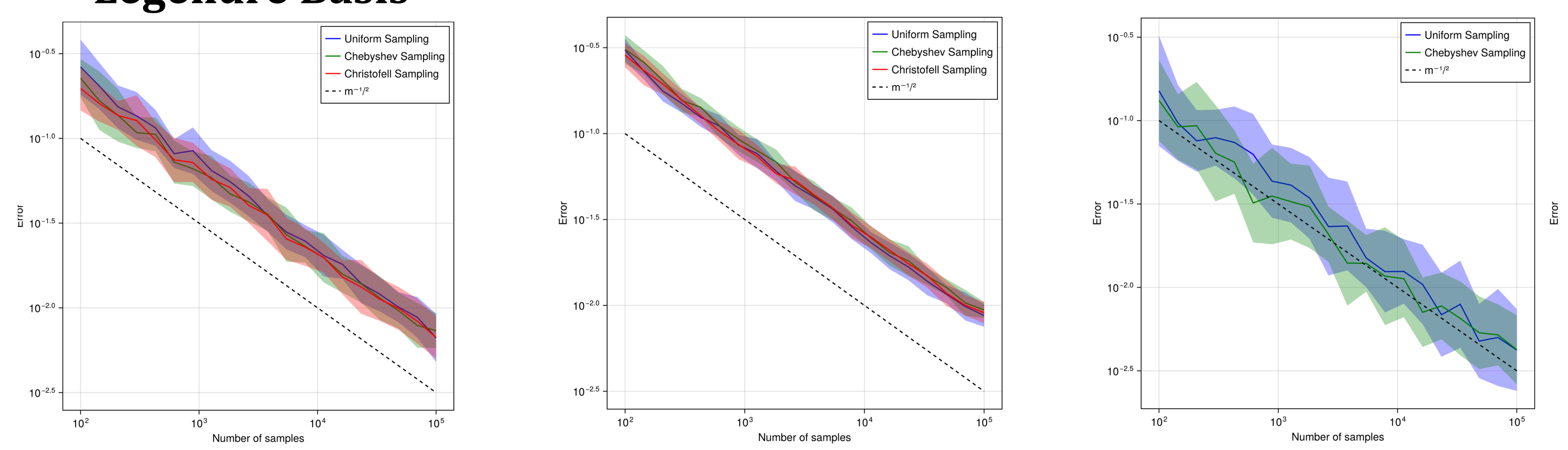
l independent least squares problem $\longleftarrow$ $\min_{(K^*)_i \in \mathbb{C}^m} \|(\boldsymbol{\Phi}_n^\tau)_i^* - \boldsymbol{\Psi}_n^*(K^*)_i\|_F^2.$

We can use the theory of least-squares to get convergence rates!

What is currently known of EDMD?

- **EDMD converges to the Koopman operator** [2]
$$\lim_{m\to\infty}\lim_{n\to\infty} \mathcal{K}_{mn}^\tau = \mathcal{K}^\tau\varphi.$$
- **Convergence rate** seems to be the Monte Carlo rate $\mathcal{O}(n^{-1/2})$.
- **Numerical experiments** confirm it. It has been proven in some specific settings (such as the logistic map $x_{n+1} = 2x_n^2 - 1$).

Legendre Basis | **Fourier Basis** | **Monomial basis**



Results

Using the theory of least-squares, we were able to **prove that rate** under the assumption of **orthonormality** of $\boldsymbol{\psi}$, and **boundedness** of $\mathcal{K}^\tau\phi_i$.

Sketch of Proof

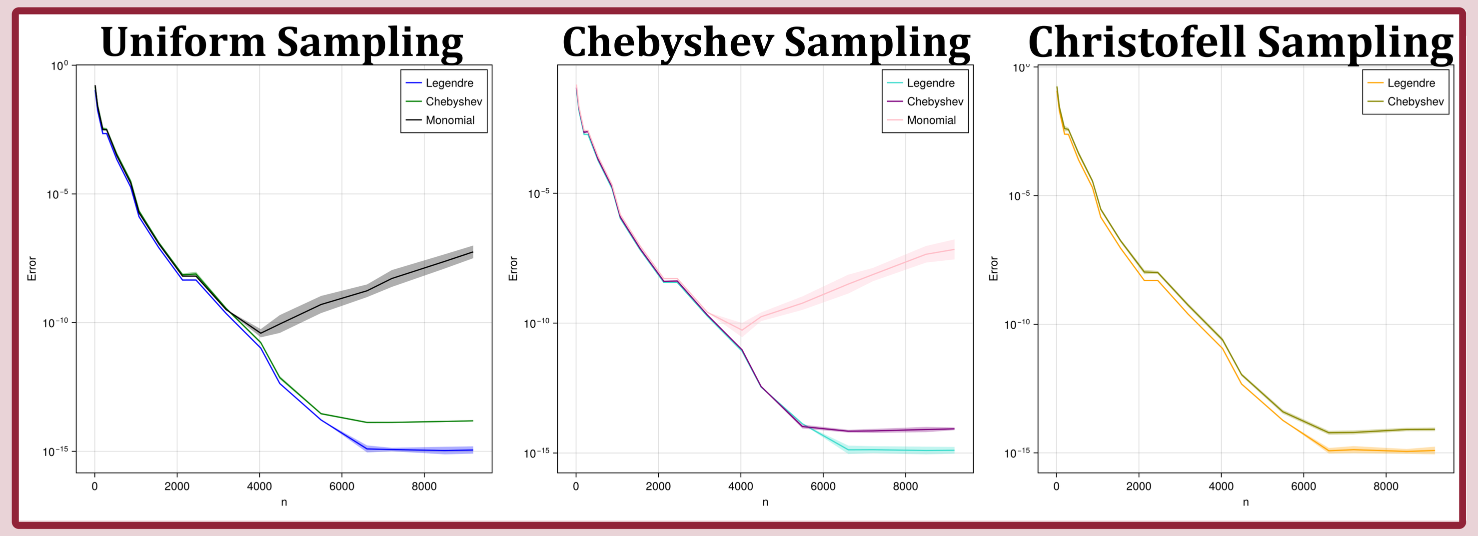
EDMD Problem $\xrightarrow{\text{Equivalently Recast}}$ **Independent least square problems** $\xrightarrow{\text{Apply least-squares theory}}$ **Derive convergence rate** $\xrightarrow{\text{Apply result to}}$ **EDMD Problem**

- **Step 1**: **Transform** the EDMD problem into independent least squares problem.
- **Step 2:** Using recovery guarantees from [3], derive a convergence rate of $\mathcal{O}(n^{-1/2})$, using **Monte Carlo approximation theory.**
- **Step 3:** Each least squares problem follows this rate. **Re-transform** into the EDMD problem to obtain the convergence rate.

$$\mathbb{E}\|\tilde{K}_{mn}^\tau - K_m^\tau\|_F^2 \leq \frac{C_1(\boldsymbol{\psi},\boldsymbol{\phi})}{n} + C_2(\boldsymbol{\phi})\epsilon \text{ with probability } 1 - l\epsilon.$$

Beyond the Monte Carlo rate

- **Recent work** [4] shows that the least squares method can converge at a super-algebraic rate.
- We numerically show that a **faster** convergence rate can be achieved if the size of the dictionary $\boldsymbol{\psi}$ grows with the number of data points, for **analytic** observables (such as the logistic map with $\varphi(x) = e^x$ below).



Discussion

We have presented new convergence results for EDMD, using minimal assumptions. These results apply to **both** the **deterministic** and **stochastic** dynamics cases.

- **Convergence rate** results for the approximation of the Koopman operator in the finite data setting are provided.
- Results are consistent with numerical experiments.
- There is still room for improvement, as our theorem assumes **orthonormality** of $\boldsymbol{\psi}$, but experiments suggest this is not necessary (see the case of monomials).
- **Compressed sensing** recovery guarantees are also derived for **LASSO-type** and **basis pursuit-type** decoders.

References

[1]: Koopman, Proc Natl Acad Sci U S A, 1931.
[2]: Bramburger, Fantuzzi, J. Nonlinear Sci., 2023
[3]: Adcock, Brugiapaglia, Webster, SIAM, 2022.
[4]: Adcock, Brugiapaglia, arXiv, 2022