Existence and Persistence of Steady States for Dynamical Systems on Large Networks

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Reaction-Diffusion on a Network

Dynamical systems on large networks are complicated since the structure of the network can make analysis challenging.

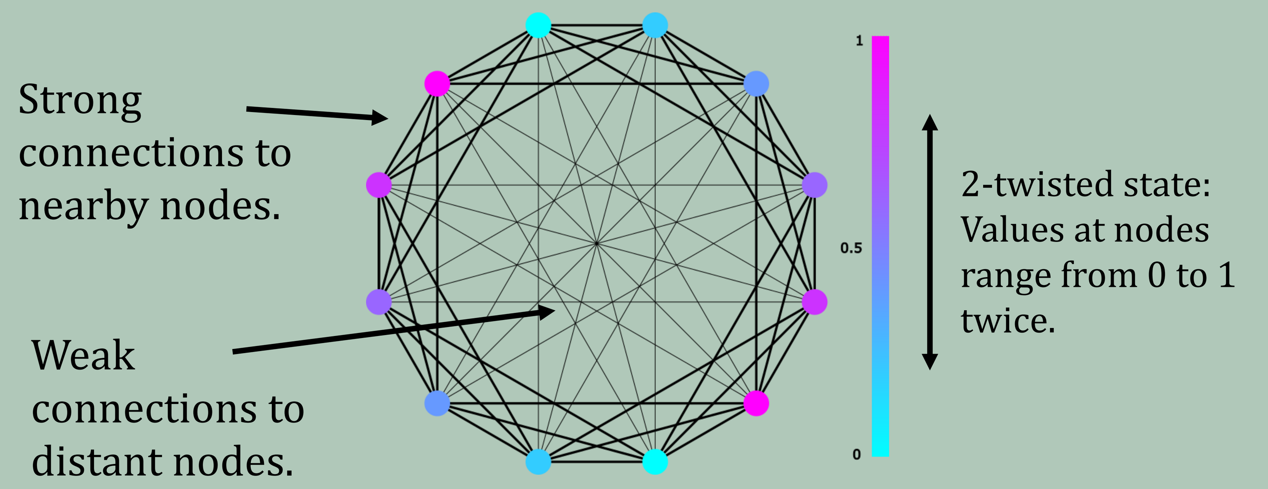
Vector in $\mathbb{R}^n$ — Local Dynamics — Adjacency Matrix — Interaction Function

$$\frac{du_i}{dt} = f_i(u_i) + \frac{1}{n}\sum_{j\neq i} A_{ij} D(u_i, u_j) \qquad (1)$$

Coupled Oscillators (Kuramoto)

$$\frac{du_i}{dt} = \frac{1}{n}\sum_{j\neq i} A_{ij}\ \sin(2\pi(u_j - u_i)) \qquad (2)$$

Let A be a small-world network and you find a family of equilibrium solutions called m-twisted states.[2][4]



Q: The strict graph structure makes this solution easy to find, but would it persist if the network were slightly changed?

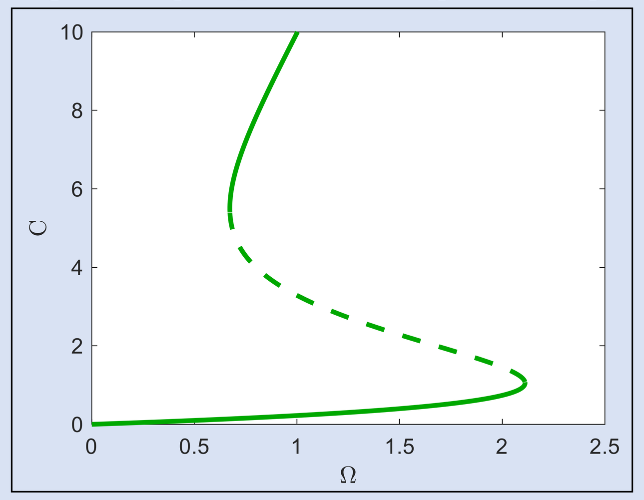
Neuron Activation (Wilson-Cowan)

Models the level of excitement in a network of neurons.[3]

$$\frac{du_i}{dt} = -u_i + \frac{\lambda}{n}\sum_{j\neq i} A_{ij}\frac{1}{1+\exp(\mu - \delta u_j)}.$$

Local damping — Nonlocal excitation

Regular graphs of degree Ω give bistability in the network.



C is a constant solution implicitly defined by:

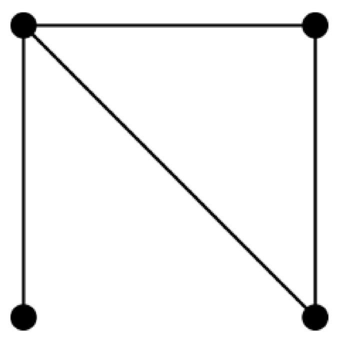
$$\frac{C}{\Omega} = \frac{\lambda}{1+\exp(\mu+\delta C)}$$

On a complete graph with edge weights Ω/n these equilibria have eigenvalues

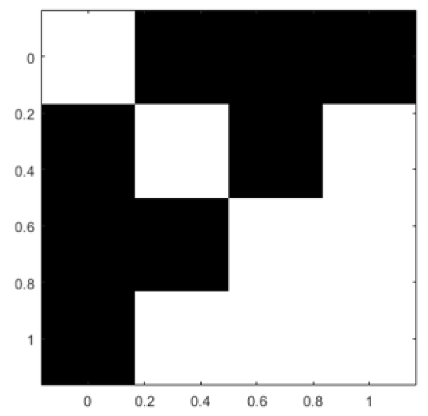
$$\lambda = -1, \qquad \frac{\Omega\left(1+\exp(\mu-\delta C)\right)^2}{\lambda\delta\exp(\mu-\delta C)} - 1.$$

Encoding Graphs as Graphons

A graphon is $W\colon [0,1]^2 \to [0,1]$ is a symmetric function that we interpret like an adjacency matrix for a graph with an uncountable vertex set. In this way we can also encode a graph as a graphon using its adjacency matrix as a step function.[1]



$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

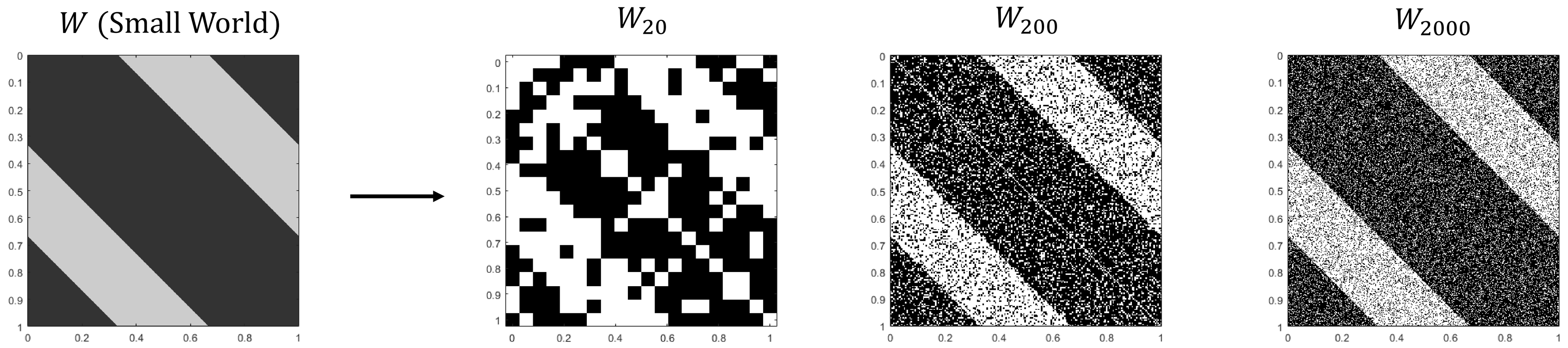


Generating Random Graphs from Graphons

Given a graphon W we generate the adjacency matrix for a random graph by

$$\mathbb{P}(A_{ij}^n = 1) = 1 - \mathbb{P}(A_{ij}^n = 0) = W\left(\frac{i}{n}, \frac{j}{n}\right) \text{ for } i > j, \qquad A_{ii}^n = 0, \qquad \text{and } A_{ji}^n = A_{ij}^n.$$

This is a matrix of Bernoulli random variables, so it is not close to W in the L^1 sense.[1]



Reaction-Diffusion on a Graphon

Replacing the adjacency matrix with a graphon yields a continuous problem

$$\frac{du}{dt}(t,x) = F(u(t,x)) = f(u(t,x)) + \int_0^1 W(x,y)D(u(t,x),u(t,y))dy. \quad (3)$$

Question

What do solutions to **(3)** tell us about solutions to the discrete system **(1)**?

Main Result

Hypothesis 1. (A sequence of random graphs) $W\colon [0,1]^2 \to [0,1]$ is a graphon and $\{A_n\}_{n\in\mathbb{N}}$ a sequence of adjacency matrices generated as above.

Hypothesis 2. (Steady state solution to $F(u)$ from **(3)**) The function $f \in C^2(\mathbb{R})$ and there exists a bounded u^* such that $F(u^*) = 0$, and $DF(u^*)$ has a bounded inverse.

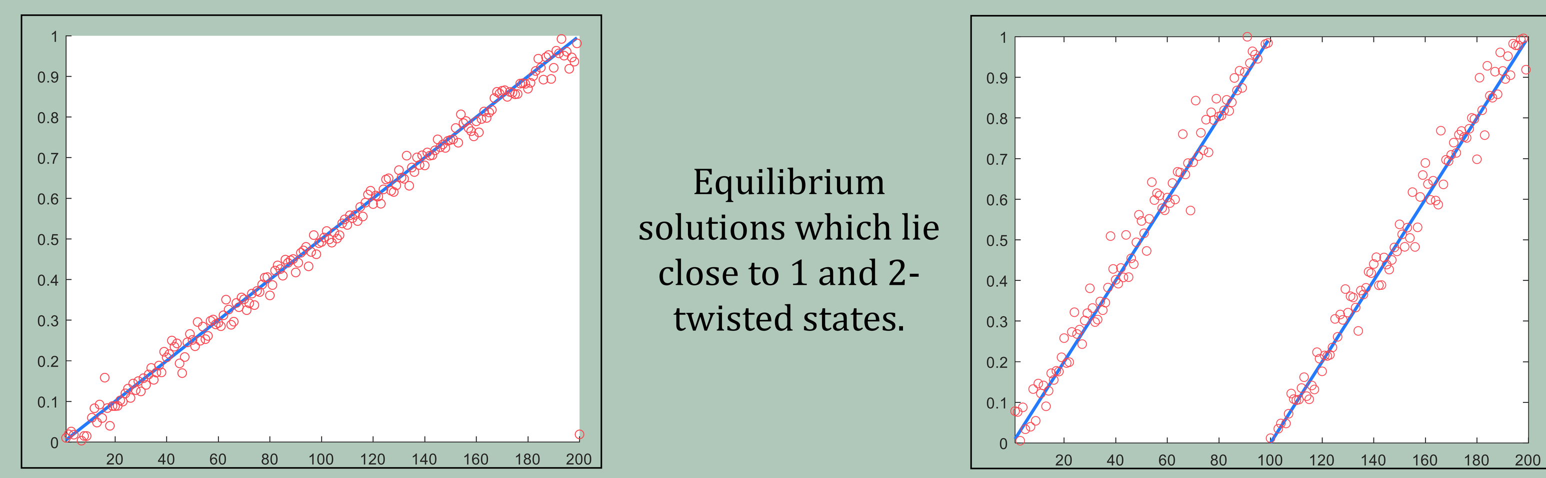
Theorem 1. (Persistence of steady state) Then there is some $N \in \mathbb{N}$ such that for $n > N$, with high probability, there is also a u^{n*}, equilibrium solution to **(1)**.

Theorem 2. (Stability of steady state) Further, there exists an $M \in \mathbb{N}$ such that for $n > \max\{N, M\}$, The equilibrium u^{n*} is stable if and only if the original equilibrium u^* is stable.

Kuramoto Model with Graphons

$$\frac{du}{dt}(x) = \int_0^1 W(x,y)\sin(2\pi(u(y) - u(x)))dy$$

For the small-world graphon W there are m-twisted equilibria again. We also find nearby equilibria using the graphon W_{200}.

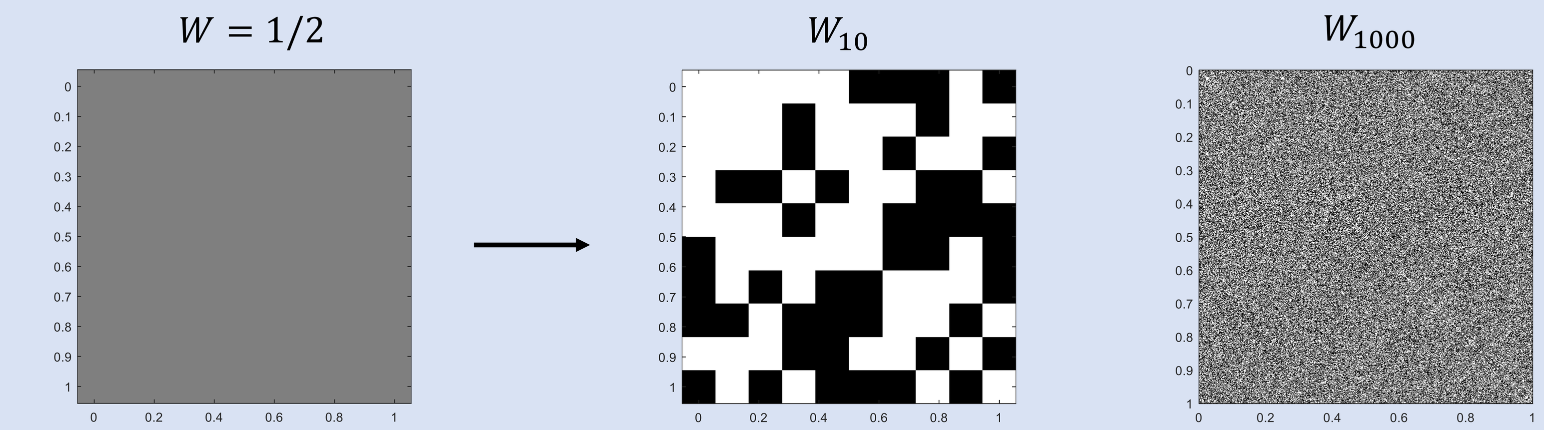


A: Yes, the solution persists even on binary graphs generated from the small-world graphon.

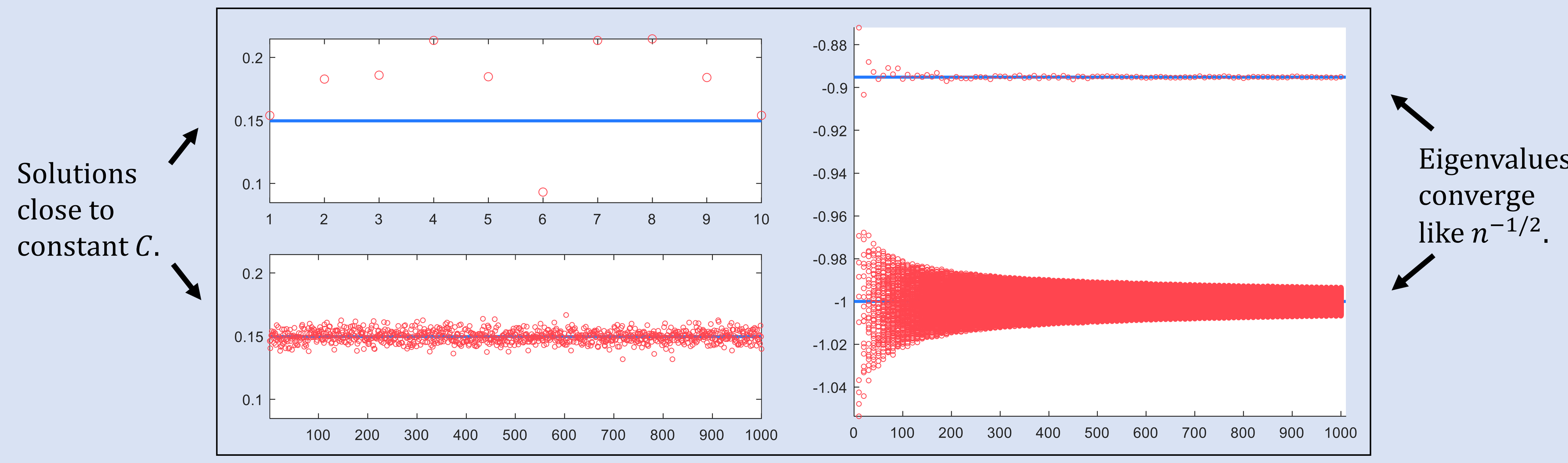
The Wilson-Cowan Model with Graphons

$$\frac{du}{dt}(x) = -u(x) + \lambda\int_0^1 W(x,y)\frac{1}{1+\exp(\mu - \delta u(y))}dy$$

We solved the discrete model on a complete graph with edge weights Ω/n. Now we start with the constant graphon $W(x,y) = \Omega$ and generate Erdős–Rényi random graphs from it.



The continuous system has a constant solution. For the discrete cases we find nearby solutions and compare their stability to the continuous case.



References

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[2] G. S. Medvedev and J. D. Wright. Stability of twisted states in the continuum Kuramoto model. SIAM J. Appl. Dyn. Syst., 16(1):188–203, 2017.
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