Snakes and Lattices: Understanding the Bifurcation Structure of Localized Solutions to Lattice Dynamical Systems

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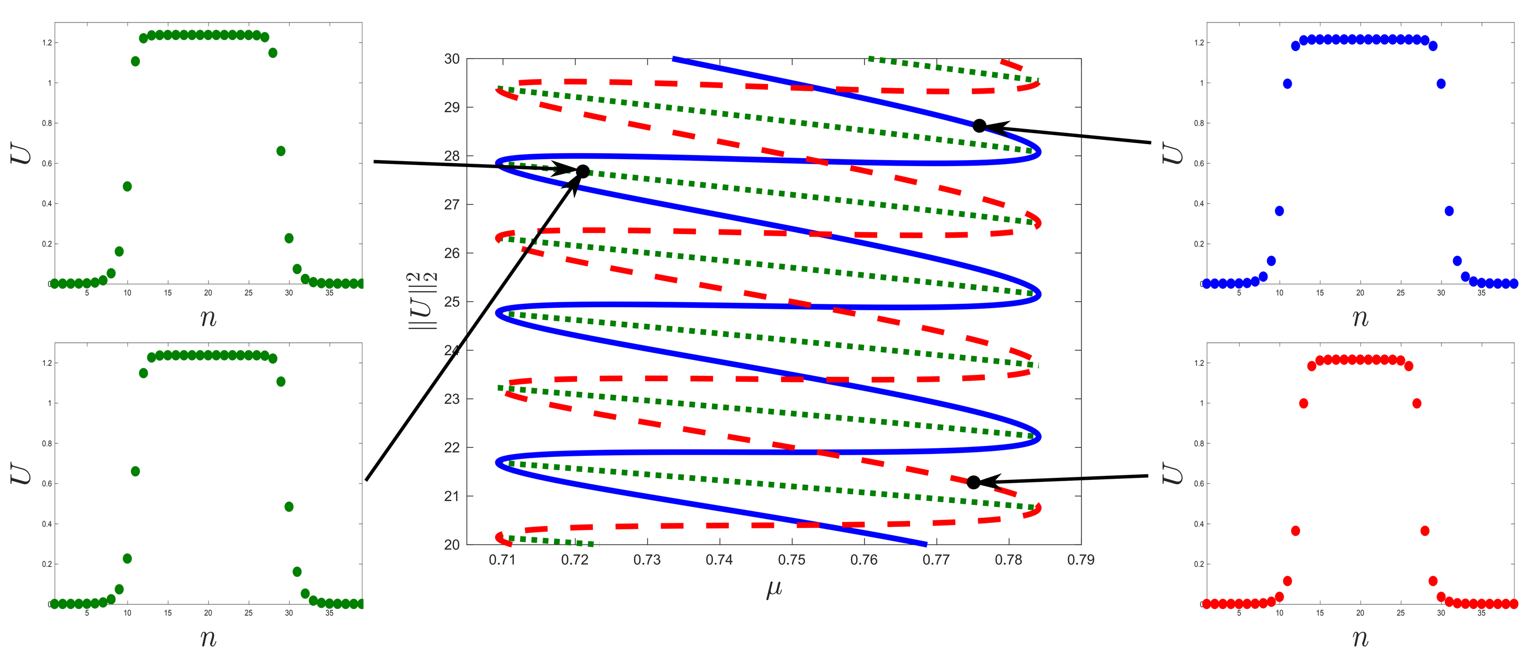
Lattice Dynamical Systems

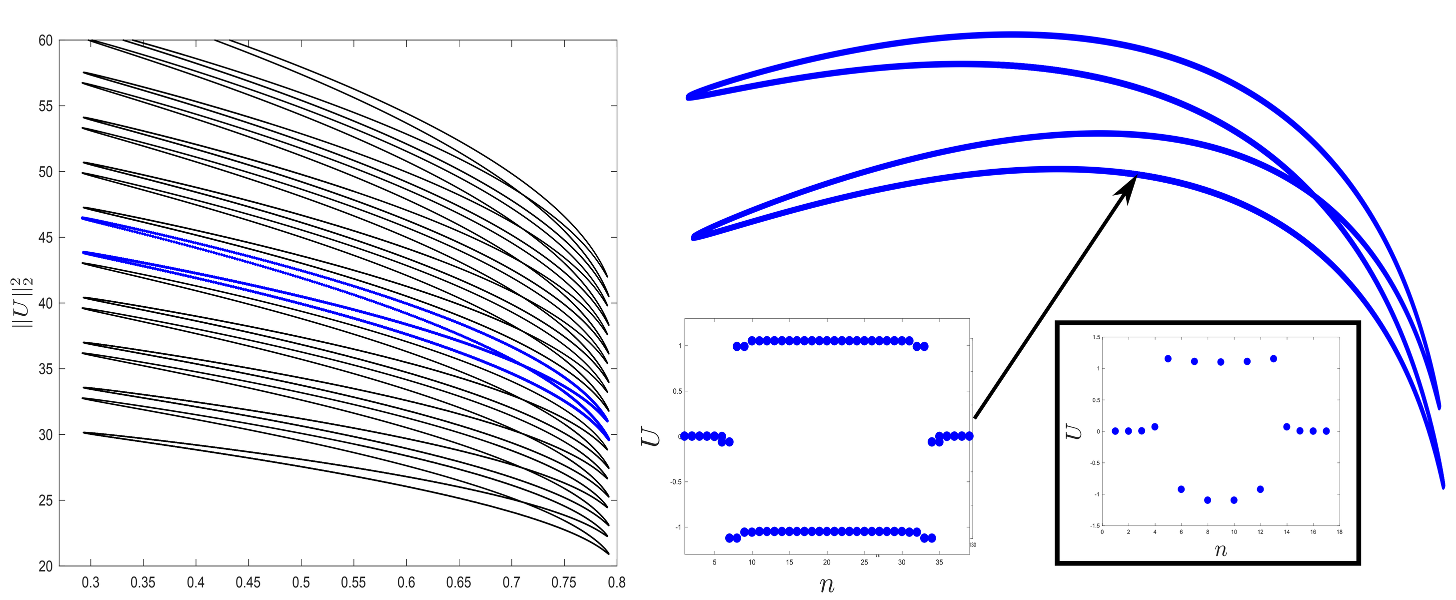
Spatially-localized structures occur in the natural world, such as in vegetation patters, crime hotspots, and ferrofluids. Taylor and Dawes [2] inspected stationary localized solutions of the lattice dynamical system

$$\dot{U}_n = d(U_{n+1} + U_{n-1} - 2U_n) - \mu U_n + 2U_n^3 - U_n^5, \quad n \in \mathbb{Z}, \quad (1)$$

where $d > 0$ represents the strength of coupling between nearest-neighbours, and μ is a bifurcation parameter. They identified solutions which lead to a snakes and ladders bifurcation diagram:



Further inspection of system (1) reveals a number of localized steady-states, including some with oscillatory plateaus. Here the bifurcation diagram does not snake, but leads to a series of stacked isolas.



Setting $\dot{U}_n = 0$ and letting $u_n = U_{n-1}$ and $v_n = U_n$ gives the discrete dynamical system

$$\begin{aligned} u_{n+1} &= v_n, \\ v_{n+1} &= 2v_n - u_n + \frac{1}{d}(\mu v_n - 2v_n^3 + v_n^5). \end{aligned} \quad (2)$$

In the map (2) localized solutions correspond to homoclinic orbits.

Acknowledgements

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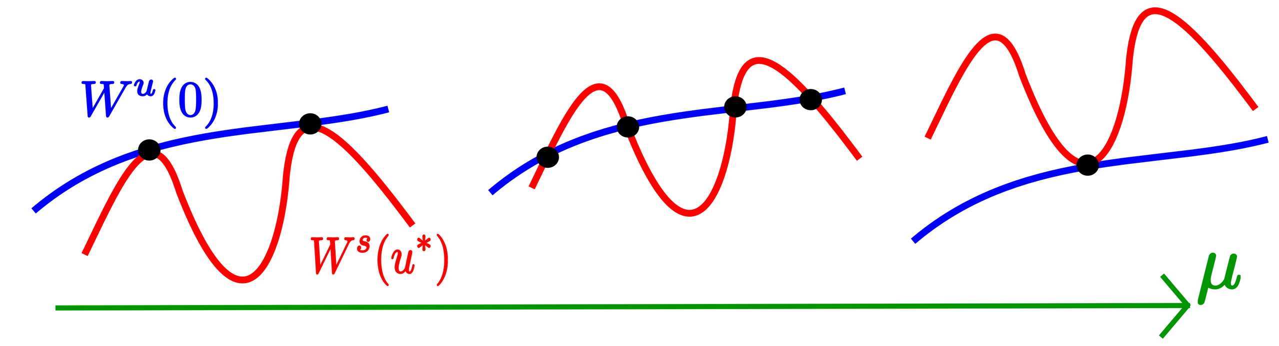
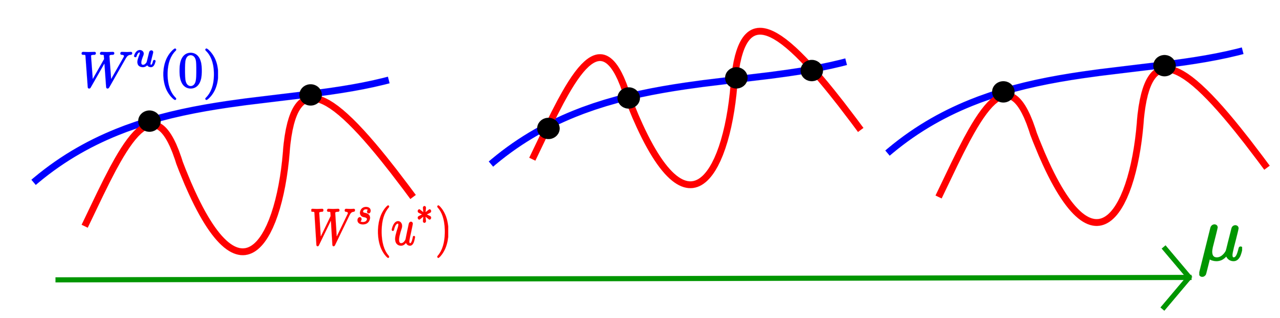
Snaking Versus Isolas

We demonstrate that the behaviour of heteroclinic orbits of (2) dictates the bifurcations of localized steady-states of (1).



Snaking is caused by intersecting stable and unstable manifolds that move through each other as μ increases.



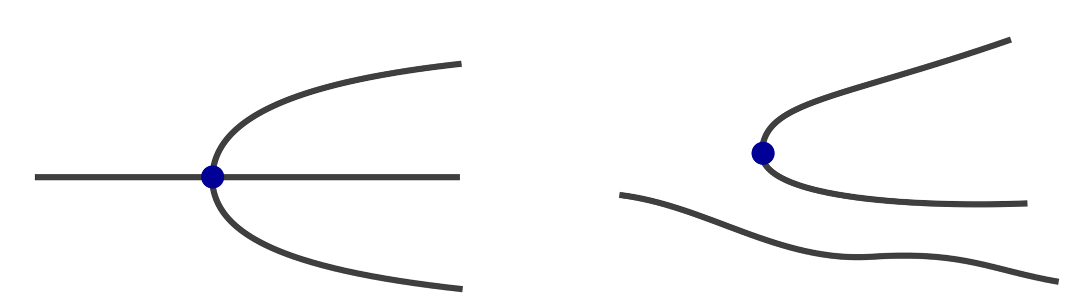
Isolas are caused by intersecting stable and unstable manifolds that do not move through each other as μ increases

The Anti-Continuum Limit

The anti-continuum limit $d = 0$ in (1) allows one to explicitly construct singular heteroclinic orbits.

∘ Away from the bifurcation points $\mu = 0, 1$ we can use the implicit function theorem to continue these heteroclinic orbits into $d > 0$

∘ Near the bifurcation points $\mu = 0, 1$ there are infinitely many bifurcations taking place. We can use singularity theory to unfold these bifurcations for small $d > 0$.



These methods allow one to explicitly determine the behaviour of the intersection of stable and unstable manifolds as μ is varied. This produces tangible affirmation of our theoretical work, something which is probably too difficult to be undertaken in the spatially continuous cases explored in [1].

References

[1] M. Beck, J. Knobloch, D. Lloyd, B. Sandstede, and T. Wagenknecht, *SIAM J. Math. Anal.* **41**, (2009) 936-972.

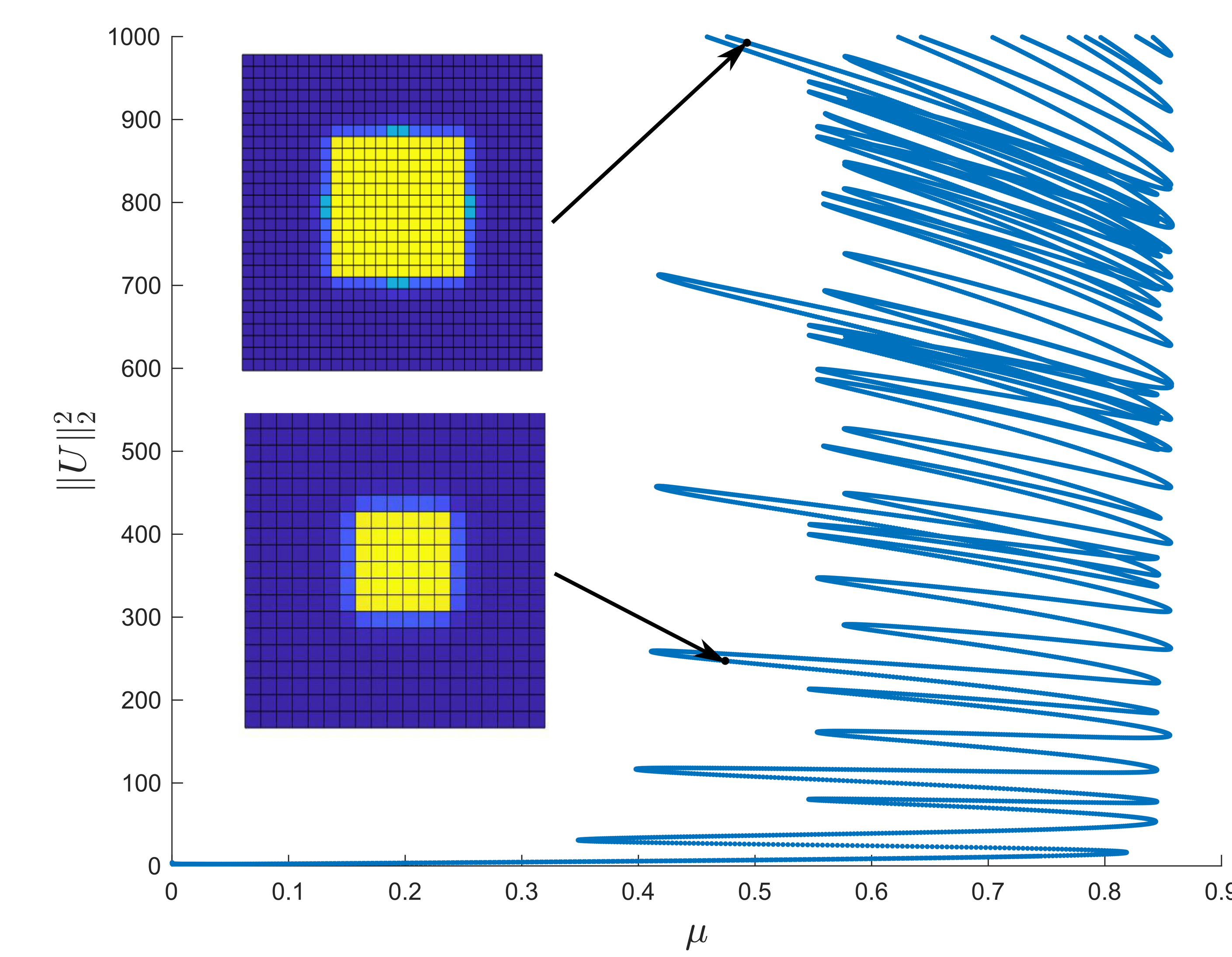
[2] C. Taylor and J.H.P. Dawes, *Phys. Rev. A* **375**, (2010) 14-22.

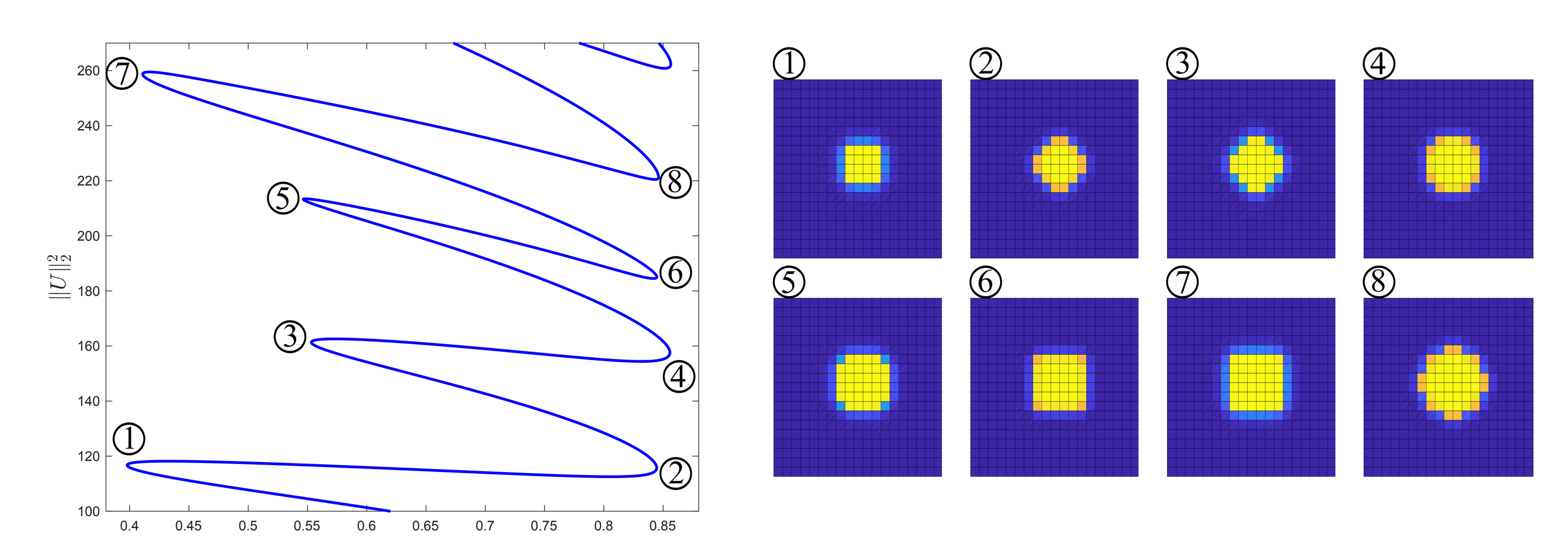
Higher Dimensional Lattices

Inspecting localized steady-states on higher dimensional lattices reveals a rich and complex bifurcation structure which is not necessarily reminiscent of the one-dimensional lattice case (1). Consider the two-dimensional analogue of (1) given by

$$\begin{aligned} \dot{U}_{n,m} = d(U_{n+1,m} + U_{n-1,m} + U_{n,m+1} + U_{n,m-1} - 4U_{n,m}) \\ - \mu U_{n,m} + 2U_{n,m}^3 - U_{n,m}^5, \quad (n,m) \in \mathbb{Z}^2. \end{aligned} \quad (3)$$

Here we find the existence of localized steady-states with a four-fold rotational symmetry, but the bifurcation structure no longer has the regular structure of the one-dimensional system:





Ongoing Work: We are optimistic that the methods of continuing solutions from the anti-continuum limit ($d = 0$) can be used to understand the irregular structure shown above.

Goal: Provide insight into the bifurcations of localized solutions in dimensions greater than one - an area that completely lacks a rigorous theoretical foundation in the PDE setting.