Snaking in the Swift-Hohenberg Equation in Dimensions $1+\varepsilon$

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The Swift-Hohenberg Equation

Spatially-localized structures occur in the natural world, such as in vegetation patters, crime hotspots, and ferrofluids. The Swift-Hohenberg equation is a widely studied nonlinear partial differential equation that can describe many spatially localized structures. Radially-symmetric solutions to the Swift-Hohenberg equation in n-dimensional space satisfy the partial differential equation

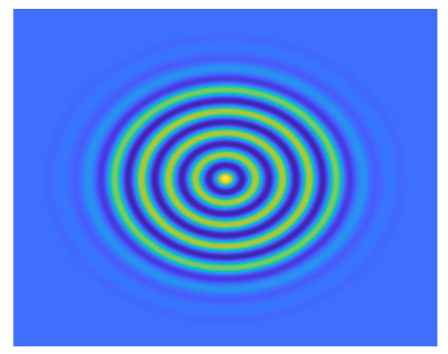


Figure 1: A localized steady-state solution to (1) for $n=2$.

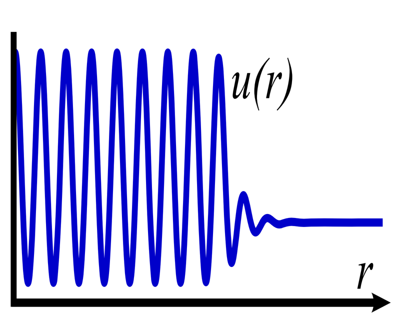
$$u_t = -\left(1+\frac{n-1}{r}\partial_r + \partial_{rr}\right)^2 u - \mu u + 2u^2 - u^3, \qquad (1)$$

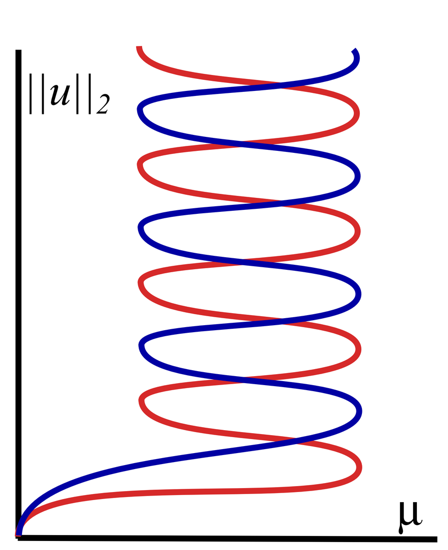
where $u = u(r,t)$, $r := |x|$, $x \in \mathbb{R}^n$, and μ is a bifurcation parameter.

The dimension of the underlying space, n, enters explicitly into equation (1). The one-dimensional equation therefore exhibits significantly different properties from the higher-dimensional Swift-Hohenberg equations:

One Dimensional Equation:	Higher Dimensional Equations:
∘ Autonomous	∘ Non-autonomous
∘ Non-Singular	∘ Singular at $r=0$
∘ Hamiltonian	∘ Not Hamiltonian

Snaking Bifurcations





In one spatial dimension ($n=1$) equation (1) possesses spatially localized pulse steady-states which exhibit a bifurcation phenomena known as *snaking* [1].

∘ Solutions of the form shown to the left bounce between two different values of the parameter μ, while ascending in the L^2-norm by simply adding another roll to the front of the wave train.

∘ It is known that these pulses come in pairs: one with a maximum at $r=0$ and another with a minimum at $r=0$.

∘ Bifurcation diagram resembles two intertwined snakes which ascend vertically in an unbounded manner.

Acknowledgements

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Snaking in Higher Dimensions

Moving to higher spatial dimensions ($n=2,3$) the bifurcation structure of the pulse steady-state solutions splits into three distinct components: a *lower snaking branch*, *isolas*, and an *upper snaking branch*.

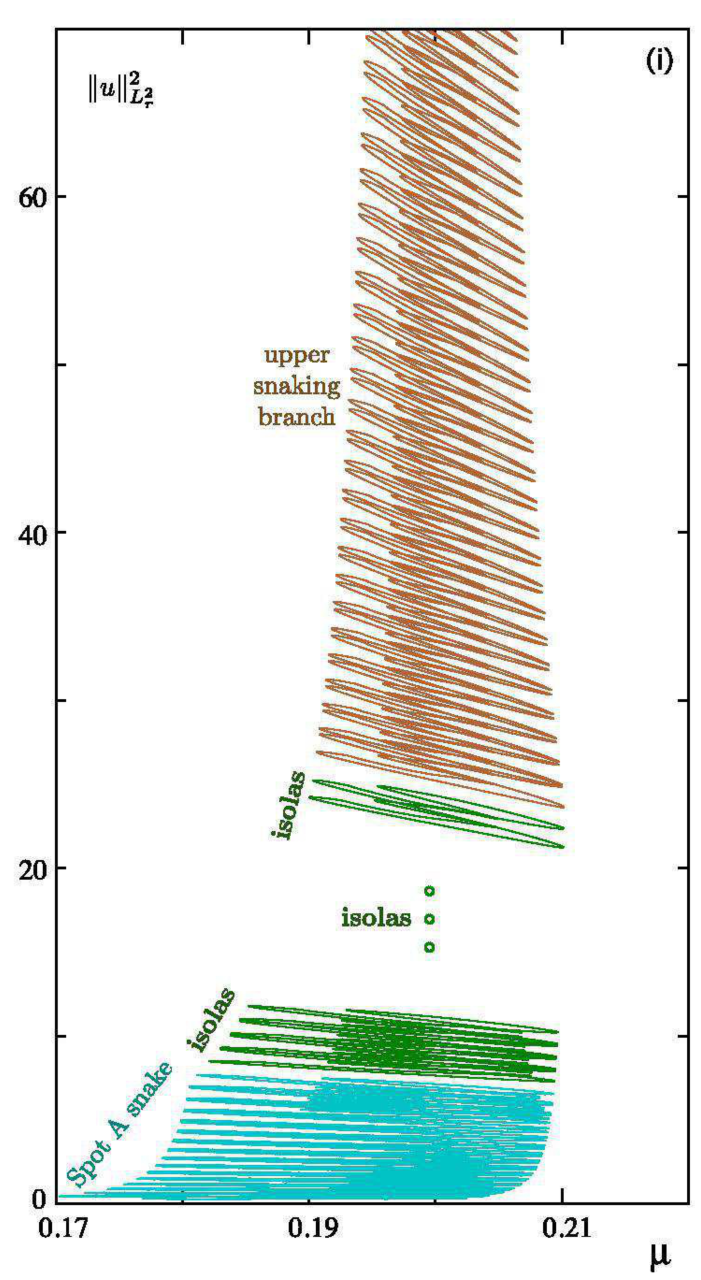


Figure 2: Image taken from [2]

Lower Snaking Branch:
∘ Bifurcation behaviour analogous to 1D equation
∘ Only extends vertically to finite height

Isolas:
∘ Collection of closed curves
∘ Start after maximum height of lower branch
∘ Only extend vertically to finite height

Upper Snaking Branch:
∘ Start after maximum height of isolas
∘ Rolls are added from the back at $r=0$
∘ Conjectured to extend infinitely in the vertical direction

Open Problems

1. What causes the lower branch to have finite height and why does it behave similar to 1D snaking?
2. What drives the formation of the isolas and the upper snaking branch?
3. Are the isolas and upper snaking branch unique to the Swift-Hohenberg equation, or should they be expected when moving to higher spatial dimensions in other reaction-diffusion type equations which exhibit snaking in 1D?

Dimensional Perturbation

To understand the higher dimensional snaking cases, we focus on introducing a dimensional perturbation into equation (1) by considering $n := 1+\varepsilon$, for small $\varepsilon > 0$. We are then able to use pertubative techniques to continuously vary ε and inspect how the non-autonomous perturbation effects the snaking bifurcation curves.

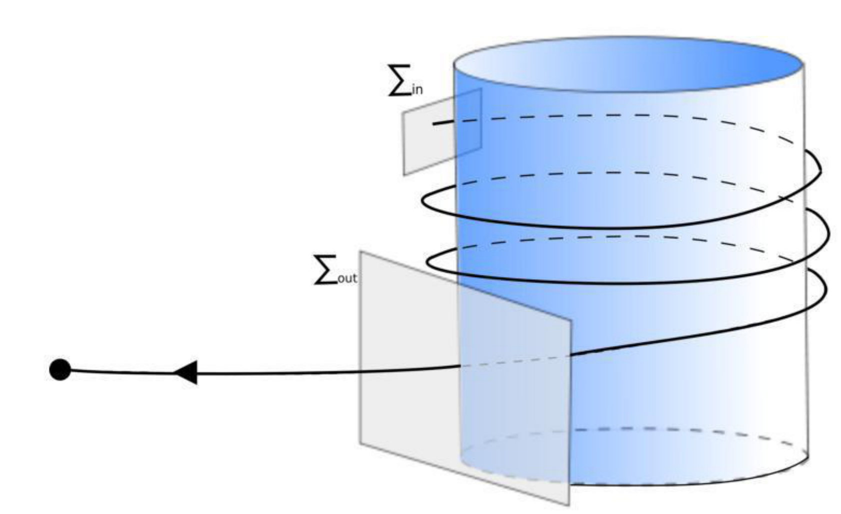
Spatial Dynamics

Steady-state solutions of (1) with this dimensional perturbation then satisfy

$$0 = -\left(1+\frac{\varepsilon}{r}\partial_r + \partial_{rr}\right)^2 u - \mu u + 2u^2 - u^3,$$

which is now a fourth-order ordinary differential equation. Letting $u_1 = u$, $u_2 = \partial_r u$, $u_3 = (1+\frac{\varepsilon}{r}\partial_r + \partial_{rr})u$ and $u_4 = \partial_r u_4$, we can consider the equivalent first order system

$$\begin{aligned}
(u_1)_r &= u_2,\\
(u_2)_r &= u_3 - u_1 - \frac{\varepsilon}{r}u_2,\\
(u_3)_r &= u_4,\\
(u_4)_r &= -u_3 - \mu u_1 + 2u_1^2 - u_1^3 - \frac{\varepsilon}{r}u_4.
\end{aligned} \qquad (2)$$



In the first order system (2) pulses correspond to solutions which start near a cylinder in phase space, spiral around it for a long period of time, and converge in forward time to the origin.

Results

∘ We are able to show that for small $\varepsilon > 0$ the lower snaking branch is formed in a similar way to the one-dimensional snaking curves, and letting L be this upper bound in L^2-norm, we find that it changes as a function of ε, and is approximately given by:

$$L = e^{\frac{1}{\varepsilon}}$$

∘ In more general PDEs, we determine sufficient conditions for the lower snaking branch to have no upper bound based upon the flow in the direction of the energy.

∘ The formation of the isolas and upper snaking branch still remain an open topic of investigation which will be the subject of future work.

References

[1] M. Beck, J. Knobloch, D. Lloyd, B. Sandstede, and T. Wagenknecht, *SIAM J. Math. Anal.* **41** (2009), 936-972.

[2] S. McCalla and B. Sandstede, *Phys. D.* **239** (2010), 1581-1592.